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# Localization of the Spectrum of Transfer Matrix of Ising Model

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The description of the whole spectrum of the multiplicative clustering operator $\mathscr{T}$ in terms of its bound states is given. Namely, it is shown that

$$\sigma(\mathscr{T}) = \sigma[\Gamma(\mathscr{T}|_b)] \qquad (0.1)$$

where $\mathscr{H}_b$ is the space of bound states of the operator $\mathscr{T}$ and $\Gamma$ is the second quantization operation.

**KEY WORDS**: Transfer matrix; clustering operator; essential spectrum; bound states.

## 1. INTRODUCTION

The $(\nu+1)$-dimensional, $\nu \geqslant 1$, Ising model is the most investigated system of statistical mechanics, but many problems concerning it remain unsolved. Among them is the problem of spectral analysis of its transfer matrix $\mathscr{T}$.

The transfer matrix was introduced by Onsager for the two-dimensional Ising model as a useful tool of investigation of a completely integrable system of statistical mechanics[(1)]. After the appearance of euclidean strategy in the quantum field theory, the transfer matrix has taken a new meaning.[(2)] Its spectrum describes the "particle structure" of the lattice model of quantum field theory. Investigations of transfer matrix from this point of view were started by Minlos and Sinai.[(3)] They gave an idea about multiplicative structure of matrix elements of $\mathscr{T}$ in a special basis. This idea was realized in Ref. 4 ($\nu = 1$) and in Ref. 5 ($\nu \geqslant 1$), where it has been proved that $\mathscr{T}$ is unitarily isomorphic to the so-called multiplicative clustering operator. For this reason, the present paper concentrates on multiplicative and additive clustering operators.

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Among the results concerning the spectrum of the operator $\mathcal{T}$ for the Ising model at high temperature we note the work of Malyshev and Minlos,[6] where the authors separate $n$-particle invariant subspaces, and the papers,[7,8] in which the absence of two-particle and three-particle bound states is proven. There exists a conjecture, connected with the above results, called a "quasi-particle picture," which says that $\mathcal{T}$ is unitarily equivalent to $\Gamma(\mathcal{T}|_{\mathcal{H}_1}))$ in $\Gamma_+(\mathcal{H}_1)$, where $\mathcal{H}_1$ is the one-particle subspace of $\mathcal{T}$ and $\Gamma_+(\mathcal{H}_1)$ is the symmetrie Fock space constructed over $\mathcal{H}_1$ (see Ref. 9).

In the present paper we shall prove one step of the "quasi-particle picture," the formula (0.1). This result is a generalization of the HVZ Theorem about the spectrum of the multiparticle Schrodinger operator (see Refs. 10,11) to multiplicative clustering operators. Equation (0.1) reduces the localization of the spectrum of $\mathcal{T}$ to finding its bound states. The next step would be the investigation of the absolute continuity of $\sigma(\mathcal{T})$ in $n$-particle subspaces. We put this off for forthcoming papers.

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## 2. NOTATIONS AND STATEMENT OF THE MAIN RESULT

Let $C_{\mathbb{Z}}{}^{\nu}$ be the set of all finite subsets of $\nu$-dimensional integer lattice $\mathbb{Z}^{\nu}$. We consider the Hilbert space

$$\mathcal{H} = l_2(C_{\mathbb{Z}}\nu)$$

with orthogonal basis $(e_T,\ T \in C_{\mathbb{Z}}\nu)$

$$e_T(T') = \delta_{T,T'}, \qquad T, T' \in C_{\mathbb{Z}}\nu$$

*Definition 2.1.* The self-adjoint operator $\mathcal{T}$ in $\mathcal{H}$ is called a *multiplicative clustering* if its matrix elements have the following expansion

$$\begin{aligned} &(e_\emptyset, \mathcal{T} e_\emptyset) = 1 \\ &(e_T, \mathcal{T} e_{T'}) = \sum_{s \geqslant 1} \sum_{\{L_i, L_i'), i=1,\dots,s\}} \prod_{i=1}^{s} \omega(L_i, L_i'), \qquad T \cup T' \neq \emptyset \end{aligned} \tag{2.1}$$

where the summation is over all (unordered) partitions $\{(L_1, L_1'),\dots, (L_s, L_s')\}$ of the pair $(T, T')$ (i.e., $\bigcup L_i = T$, $\bigcup L_i' = T'$, $L_i \cap L_j = L_i' \cap L_j' = \emptyset$ for $i \neq j$), such that

$$L_i \neq \emptyset \qquad \mathrm{L}_{\mathrm{i}}' \neq \emptyset \qquad \mathrm{i} = 1,\dots, s$$

The (clustering) functions $\omega(L, L') = \omega^{\mathscr{T}}(L, L')$ satisfy the following properties

(a) $\omega(L+x, L'+x) = \omega(L, L')$ $x \in \mathbb{Z}^\nu$ where $\{x_1,..., x_n\} + x = \{x_1 + x,..., x_n + x\}$ (translational invariance)

(b) $\omega(L, L') = \overline{\omega(L', L)}$ (self-adjointness)

(c)

$$0 < \omega(\{0\}; \{0\}) = \lambda < 1 \quad \text{and} \quad |\omega(L, L')| \leqslant M\lambda(1-\lambda)\,\beta^{d_{L \cap L'}} \quad \text{if} \quad |L \cup L'| \geqslant 2 \tag{2.2}$$

where $M > 0$, $0 < \beta < 1$ are constants, $|B|$ denotes the cardinality of the set $B \subset \mathbb{Z}^\nu$ and $d_B$ is the minimum length of the tree graph connecting points of the set B. (The metric in $\mathbb{Z}^\nu$ is given by the formula

$$d(x_1, x_2) = \sum |x_1^{(i)} - x_2^{(i)}|, \qquad x_j = (x_j^{(1)},..., x_j^{(1)}), j = 1, 2)$$

*Remark 2.1.* (a) The authors (see Refs. 4, 6) usually use the following stronger than (2.2) estimate

$$|\omega(L, L')| \leqslant M\beta^{d_{L \times \{0\} \cup L' \times \{t\}}}$$

which is satisfied for transfer matrix $\mathscr{T}_t$ of the Ising model (with continuous or discrete time) when $t$ is greater than certain constant $t_0 > 0$. The bound (2.2) is sufficient for purposes of this work and contains the cases of great and small $t$.

(b) There exists a notion of *additive clustering operator H*, which appears as the generator (Hamiltonian) of the semigroup $\mathscr{T}_t$ of transfer matrices of the $(\nu+1)$-dimensional Ising model with continuous time.[12] Its matrix elements satisfy the expansion

$$(e_T, He_{T'}) = \sum_{\substack{\emptyset \neq L \subset T, \emptyset \neq L' \subset T' \\ T \setminus L = T' \setminus L'}} \omega(L, L') \tag{2.3}$$

where the functions $\omega = \omega^H$ satisfy the following properties

(i) $\omega(L, L') = \omega(L+x, L'+x)$ $x \in \mathbb{Z}^\nu$

(ii) $\omega(L, L') = \overline{\omega(L', L)}$

(iii) $\omega(\{0\}, \{0\}) = a > 0$ and $|\omega(L, L')| \leqslant M\beta^{d_{L \cup L'}}$ if $|L \cup L'| \geqslant 2$ (2.4)

where $M > 0$ and $0 \leqslant \beta < 1$ are constants not depending on $L, L'$.

It turns out that $e^{-tH}$, $t \geqslant 0$, is a multiplicative clustering operator with parameters $\lambda = e^{-at}$ and $\beta_1 \to 0$ as $\beta \to 0$ (see Ref. 13). Therefore, the analysis of the spectrum of $H$ is equivalent to the analysis of the spectrum of the multiplicative operator $e^{-tH}$ for $t > 0$.

(c) The definition of general clustering operators and their basic properties are given in Ref. 12.

We recall now certain known results about the clustering operators and introduce some notations. The letter $A$ will denote either $\mathscr{T}$ or $H$. Denote

$$\begin{aligned} &L_n = \{f \in \mathscr{H} : f(T) = 0 \quad \text{if} \quad |T| \neq n\} \\ &\mathscr{L}^{(0)} = L_0 \qquad \mathscr{L}^{(1)} = \mathscr{H} \ominus L_0 \\ &A^{(0)} = A\,|_{\mathscr{L}^{(0)}} \qquad A^{(1)} = A\,|_{\mathscr{L}^{(1)}} \end{aligned} \tag{2.5}$$

In the space $\mathscr{L}^{(1)}$ acts the group $\mathbb{Z}^\nu$ by translations

$$U_x e_T = e_{T-x} \qquad x \in \mathbb{Z}^\nu \qquad T \in C_{\mathbb{Z}^\nu}$$

The operator $A$ commutes with this action and hence we have the following direct integral representation

$$\mathscr{L}^{(1)} = \int_{\mathbb{T}^\nu \oplus} \mathscr{L}^{(1)}_\Lambda \, d\Lambda \qquad A^{(1)} = \int_{\mathbb{T}^\nu \oplus} A^{(1)}_\Lambda \, d\Lambda \tag{2.6}$$

where $\mathbb{T}^\nu$ is $\nu$-dimensional tori and

$$\begin{aligned} \mathscr{L}^{(1)}_\Lambda = \Bigg\{ f : f(T+x) = e^{2\pi i (x,\Lambda)} f(T) \qquad \|f\|^2_{\mathscr{L}^{(1)}_\Lambda} \\ = \sum_{T:0 \in T} \frac{1}{|T|} |f(T)|^2 < \infty \Bigg\} \\ A^{(1)}_\Lambda = A\,|_{\mathscr{L}^{(1)}_\Lambda} \end{aligned}$$

Denote also for $\Lambda \in \mathbb{T}^\nu$

$$\begin{aligned} &\mathscr{L}^{(2)}_\Lambda = \int_{\mathbb{T}^\nu \oplus} \mathscr{L}^{(1)}_\lambda \otimes \mathscr{L}^{(1)}_{\Lambda - \lambda} \, d\lambda \\ &\mathscr{T}^{(2)}_\Lambda = (\mathscr{T}^{(1)} \otimes \mathscr{T}^{(1)})\,|_{\mathscr{L}^{(2)}_\Lambda} \\ &H^{(2)}_\Lambda = (H^{(1)} \otimes id + id \otimes H^{(1)})\,|_{\mathscr{L}^{(2)}_\Lambda} \end{aligned} \tag{2.7}$$

**Theorem 1.** Let $\mathscr{T}$ be the multiplicative clustering operator with parameters $\lambda$ and $\beta$. Then for $\lambda$ sufficiently small and for any $\Lambda \in \mathbb{T}^\nu$

$$\sigma_{ess}(\mathscr{T}^{(1)}_\Lambda) = \sigma(\mathscr{T}^{(2)}_\Lambda) \tag{2.8}$$

(Here $\sigma$ ($\sigma_{ess}$ or $\sigma_d$) denotes spectrum (essential spectrum or discrete spectrum)).

For $A = \mathscr{T}$ or $H$ we denote $\mathscr{H}_1 =$ one-particle invariant subspace (see Refs. 3, 6)

$$\sigma_b(A^{(1)}) = \bigcup_{\Lambda \in \mathbb{T}^\nu} \sigma_d(A_\Lambda^{(1)})$$
$$\mathscr{H}_b = \int_{\mathbb{T}^\nu \oplus} \mathscr{H}_{b,\Lambda}\, d\Lambda \tag{2.9}$$

where $\mathscr{H}_{b,\Lambda} =$ span {eigenvectors with eigenvalues of finite multiplicity of $A_\Lambda^{(1)}$}.

Consider now the space

$$\mathscr{B} = \Gamma(\mathscr{H}_b)$$

In this space acts the group $\mathbb{Z}^\nu$ via the translations $U_x$. Hence

$$\mathscr{B} = \int_{\mathbb{T}^\nu \oplus} \mathscr{B}_\Lambda\, d\Lambda$$

where

$$\mathscr{B}_\Lambda = \mathbb{C}\delta_{\Lambda,0} \oplus \bigoplus_{n=0}^{\infty} \int_{\Lambda_1 + \cdots + \Lambda_n = \Lambda} \mathscr{H}_{b,\Lambda_1} \otimes \cdots \mathscr{H}_{b,\Lambda_n}$$

In the space $\mathscr{B}$ acts the operator $D = \Gamma(\mathscr{T} \mid \mathscr{H}_b) = \int D_\Lambda d_\Lambda$ commuting with the action of $\mathbb{Z}^\nu$. Therefore we have an infinite particle system ($\mathscr{B}$, $U_x$, $D^{it}$) homogenous with respect to the translations and with dynamics defined by the operators $D^{it}$, $t \in \mathbb{R}$. From Theorem 1 follows the following result.

**Theorem 2.** The joint spectrum (a subset of $\mathbb{T}^\nu \times \mathbb{R}^1$) of translations and transfer-matrix of the Ising model, is the same as that one in the above free system.

Theorem 2 has some obvious consequences.

**Corollary 2.1.** (a) The spectrum of $\mathscr{T}$ forms a semigroup

$$\sigma(\mathscr{T}) = \{0\} \cup \bigcup_{n=0}^{\infty} (\sigma_b(\mathscr{T}^{(1)}))^n = \sigma(D)$$

(b) $\sigma(\Gamma(\mathscr{T}\,|_{\mathscr{H}_1})) \subset \sigma(\mathscr{T})$

(c) $\sigma(H) = \bigcup_{n=0}^{\infty} \sum_{i=1}^{n} \sigma_b(H^{(1)}) = \sigma(d\Gamma(H\,|_{\mathscr{H}_b}))$

(d) $\sigma(d\Gamma(H\,|_{\mathscr{H}_1})) \subset \sigma(H)$

The property (d) of Corollary 2.1 for $H$ (a Hamiltonian of the Ising model with continuous time) was proved by Malyshev in Ref. 9 with the help of the scattering theory methods. Corollary 2.1 gives the description of the spectrum of the operator $\mathscr{T}$ (or $H$) in terms of thresholds as in the HVZ Theorem about the essential spectrum of the Schrodinger operator (see Refs. 10, 11). This makes possible the construction of the scattering theory for clustering operators.

Theorems 1 and 2 can be easily generalized to the lattice systems with bounded spin in the domain of existence of exponentially regular cluster expansion. The author hopes that some analogies of Theorems 1 and 2 hold for unbounded spin lattice systems and for continuous $P(\varphi)_2$ quantum field theory.

Proof of Theorem 1 consists of two parts corresponding to the inclusions

$$\sigma(\mathscr{T}_\Lambda^{(2)}) \subset \sigma_{\text{ess}}(\mathscr{T}_\Lambda^{(1)}) \qquad \text{and} \qquad \sigma_{\text{ess}}(\mathscr{T}_\Lambda^{(1)}) \subset \sigma(\mathscr{T}_\Lambda^{(2)})$$

and is given in the next two sections. The proof of Theorem 2 we put off to the last section.

## 3. PROOF OF THE INCLUSION $\sigma(\mathscr{T}_\Lambda^{(2)}) \subset \sigma_{\text{ess}}(\mathscr{T}_\Lambda^{(1)})$

Since $\sigma_{\text{ess}}(\mathscr{T}_\Lambda^{(2)}) = \sigma(\mathscr{T}_\Lambda^{(2)})$ (see (2.7)) it is enough to prove the inclusion

$$\sigma(\mathscr{T}_\Lambda^{(2)}) \subset \sigma(\mathscr{T}_\Lambda^{(1)}) \tag{3.1}$$

The idea in the proof of (3.1) is simple. If $\lambda \in \sigma(\mathscr{T}_\Lambda^{(2)})$ then one can choose the approximate eigenfunction of $\mathscr{T}_\Lambda^{(2)}$ of the form $\varphi_1(T_1) \cdot \varphi_2(T_2)$, where $\varphi_i \in \mathscr{L}_{\Lambda_i}^{(1)}$, $i = 1, 2$, $(\Lambda_1 + \Lambda_2 = \Lambda)$, are approximate eigenfunctions of $\mathscr{T}_{\Lambda_i}^{(1)}$ with eigenvalues $\lambda_i$, $\lambda_1 \lambda_2 = \lambda$. The function $\varphi(T) = \sum \varphi_1(T_1)\, \varphi_2(T_2 + t)$, where we sum over such $T_1$, $T_2$ that $T = T_1 \cup (T_2 + t)$ and $|t|$ is large is an approximate eigenfunction of $\mathscr{T}_\Lambda^{(1)}$ with eigenvalue $\lambda$. This is due to the independence (clustering) of the matrix elements of $\mathscr{T}$ at large distances (see 2.2).

In order to be precise we define a map

$$G : \mathscr{L}^{(1)} \otimes \mathscr{L}^{(1)} \to \mathscr{L}^{(1)}$$

$$G\varphi(T) = \sum_{\substack{T_1 \neq \emptyset, T_2 \neq \emptyset \\ T_1 \cup T_2 = T \\ T_1 \cap T_2 = \emptyset}} \varphi(T_1, T_2), \qquad \varphi \in \mathscr{L}^{(1)} \otimes \mathscr{L}^{(1)}, \qquad T \in C_{\mathbb{Z}^\nu} \tag{3.2}$$

**Lemma 3.1.** The restriction of $G$ onto $\mathscr{L}_\Lambda^{(2)}$ defines the (unbounded) operator $G:\mathscr{L}_\Lambda^{(2)}\to\mathscr{L}_\Lambda^{(1)}$.

*Proof.* Any element of $\mathscr{L}_\Lambda^{(2)}$ can be represented as a function $\varphi$ of two variables such that

$$\varphi(T_1+x, T_2+x)=e^{2\pi i(x,\Lambda)}\varphi(T_1, T_2) \qquad T_1, T_2\in C_{\mathbb{Z}^\nu} \qquad x\in\mathbb{Z}^\nu$$

and (3.3)

$$\|\varphi\|^2_{\mathscr{L}_\Lambda^{(2)}}=\sum_{T_1,T_2:0\in T_1,T_2\neq\emptyset}\frac{1}{|T_1|}|\varphi(T_1, T_2)|^2<\infty$$

From this Lemma 3.1 follows.

We introduce also the unitary operators

$$S_t:\mathscr{L}^{(1)}\otimes\mathscr{L}^{(1)}\to\mathscr{L}^{(1)}\otimes\mathscr{L}^{(1)}, \qquad t\in\mathbb{Z}^\nu$$

by the formula

$$S_t\varphi(T_1, T_2)=\varphi(T_1, T_2+t) \qquad \varphi\in\mathscr{L}^{(1)}\otimes\mathscr{L}^{(1)} \qquad T_1, T_2\in C_{\mathbb{Z}^\nu} \tag{3.4}$$

**Lemma 3.2.** The operators $S_t$ leave the expansion

$$\mathscr{L}^{(1)}\otimes\mathscr{L}^{(1)}=\iint\mathscr{L}_\lambda^{(1)}\otimes\mathscr{L}_\mu^{(1)}\,d\lambda\,d\mu$$

invariant and commute with the operator $\mathscr{T}^{(1)}\otimes\mathscr{T}^{(1)}$.

*Proof.* It is a simple consequence of definitions.

Assume that $\lambda\in\sigma(\mathscr{T}^{(2)})$. Then for every $\varepsilon>0$ there exists $\psi\in\mathscr{L}_\Lambda^{(2)}$, $\|\psi\|=1$, such that

$$\|(\mathscr{L}_\Lambda^{(2)}-\lambda)\,\psi\|\leqslant\varepsilon \tag{3.5}$$

We can assume that $\psi$ is finite in the sense that the set

$$\{(T_1, T_2):0\in T_1, \psi(T_1, T_2)\neq 0\}$$

is finite. Let us define

$$\psi_t=GS_t\psi\in\mathscr{L}_\Lambda^{(1)} \tag{3.6}$$

We show that for large $|t|$ $\psi_t\neq 0$ and $\psi_t$ is an approximate eigenfunction of the operator $\mathscr{L}_\Lambda^{(1)}$ with eigenvalue $\lambda$ (see Lemma 3.3 and 3.4 below). This is enough to the proof of the inclusion (3.1).

**Lemma 3.3.** If $|t|$ is sufficiently large then $\|\psi_t\|_{\mathscr{L}_\Lambda^{(1)}}\geqslant 1$.

*Proof.* Let us fix a certain order for each set $T \subset \mathbb{Z}^\nu$ (lexicographic, for example). Then we obtain

$$\psi_t(T) = \sum_{\substack{T_1 \neq \emptyset, T_2 \neq \emptyset \\ T_1 \cup T_2 = T \\ T_1 \cap T_2 = \emptyset}} \psi(T_1, T_2 + t) = \sum_\alpha \psi^{(\alpha)}(T) \tag{3.7}$$

where each $\psi^{(\alpha)}$ corresponds to the different partition $\alpha$ of the set $\{1,..., n(\alpha)\}$, $n(\alpha) = |T| = 2, 3,...,$ into two subsets. Because $\psi$ is finite $\psi^{(\alpha)}$ have nonintersecting supports for large $|t|$ and $\|\psi^{(\alpha)}\|_{\mathscr{L}_A^{(1)}} = \|P_{n(\alpha)}\psi\|_{\mathscr{L}_A^{(2)}}$, where $P_n$ is the orthogonal projection onto $L_n$ (see (2.5)).

**Lemma 3.4.** We have an estimate

$$\|(\mathscr{T}^{(1)} - \lambda)\, \psi_t\|_{\mathscr{L}_A^{(1)}} \leqslant 3\varepsilon$$

for large $|t|$.

*Proof.* By (2.1), (3.3), and (3.4) we have

$$[(\mathscr{T}^{(1)} - \lambda)\, \psi_t](T)$$
$$= \sum_{T' \neq \emptyset} \left\{ \left[ \sum_{s \geqslant 1} \sum_{\{(L_i, L_i')\}} \prod_1^s \omega(L_i, L_i') \right] - \lambda \delta_{T,T'} \right\}$$
$$\cdot \left\{ \sum_{T_1' \cup T_2' = T'} S_t \psi(T_1', T_2') \right\} = N_1(T) + N_2(T) + N_3(T)$$

where

$$N_1(T) = \sum_{\substack{T_1' \neq \emptyset, T_2' \neq \emptyset \\ T_1' \cap T_2' = \emptyset}} \sum_{\{(L_i, L_i')\}}' \prod \omega(L_i, L_i')\, \psi(T_1', T_2' + t) \tag{3.8}$$

$$N_2(T) = - \sum_{\substack{T_1 \neq \emptyset, T_2 \neq \emptyset \\ T_1 \cup T_2 = T \\ T_1 \cap T_2 \neq \emptyset}} \sum_{\substack{T_1' \neq \emptyset, T_2' \neq \emptyset \\ T_1' \cap T_2' \neq \emptyset}} \sum_{\substack{\{(L_1, L_i')\} \\ \text{partition} \\ \text{of } (T_1, T_1')}} \sum_{\substack{\{(\tilde{L}_i, \tilde{L}_i')\} \\ \text{partition} \\ \text{of } (T_2, T_2')}}$$
$$\times \left\{ \prod \omega(L_i, L_i') \prod \omega(\tilde{L}_i, \tilde{L}_i') \right\} \psi(T_1', T_2' + t) \tag{3.9}$$

$$N_3(T) = \sum_{\substack{T_1 \neq \emptyset, T_2 \neq \emptyset \\ T_1 \cup T_2 = T \\ T_1 \cap T_2 = \emptyset}} [(\mathscr{T}^{(2)} - \lambda)\, S_t \psi](T_1, T_2) \tag{3.10}$$

where the summation $\sum'$ in $N_1(T)$ is over such partitions $\{(L_i, L_i')\}$ of the pair $(T, T_1' \cup T_2')$ that $L_i' \cap T_1' \neq \emptyset$ and $L_i' \cap T_2' \neq \emptyset$ for certain $i$. Proof of Lemma 3.4 will be ended if we prove the following

**Lemma 3.5.** We have

(a) $\|N_1\| \to 0$ as $|t| \to \infty$

(b) $N_2 = 0$ for large $|t|$

(c) $\|N_3\| \leqslant 2\varepsilon$ for large $|t|$

*Proof.* (a) We change the order of summation in (3.8): we first sum over $T' \neq \emptyset$, next over partitions $\{(L_i, L_i')\}$ of the pair $(T, T')$ and at the end over partitions $(T_1', T_2')$ of the set $T'$, which satisfy the condition

$$L_i' \cap T_1' \neq \emptyset \quad \text{and} \quad L_i' \cap T_2' \neq \emptyset \quad \text{for certain } i \tag{3.11}$$

The number of partitions $(T_1', T_2')$ is bounded by a constant $D$ depending only on $\psi$ (because of finiteness of $\psi$). Hence we obtain

$$|N_1(T)| \leqslant D \sum_{T': 1 \leqslant |T'| \leqslant n} \sum_{\{(L_i, L_i')\}} \prod |\omega(L_i, L_i')| \cdot \sup_{T_1', T_2'} |\psi(T_1', T_2' + t)| \tag{3.12}$$

where $n = n(\psi)$ and sup is over partitions $(T_1', T_2')$ of the set $T'$ satisfying (3.11). We choose a number $i$, for which the condition (3.11) is true and denote $L'' = L_i' \cap T_1'$, $L''' = L_i' \cap T_2'$. By (2.2) we have

$$|\omega(L_i, L_i')| \leqslant M\lambda(1-\lambda)\, \beta^{d_{K_i \cup L_i'}} \leqslant M_1 \beta^{(1/2)|t| + (1/6) d_{L_i \cup L'' \cup (L''' + t)}} \tag{3.13}$$

One can prove the inequality (3.13) using the fact that $\psi$ is finite. The pair $(T_1', T_2')$ is such that $T_2' + t$ lies near $T_1'$ and then $L'''$ is far from $L''(d_{L_i \cup L'' \cup L'''} > |t| - \text{const})$. Considering the cases: $d_{L_i \cup L'' \cup (L''' + t)} \leqslant 3|t|$ (here $d_{L_i \cup L'' \cup L'''} \geqslant \frac{1}{2}|t| + \frac{1}{6} d_{L_i \cup L'' \cup (L''' + t)} + \text{const}$), and $d_{L_i \cup L'' \cup (L''' + t)} > 3|t|$ (here $d_{L_i \cup L'' \cup L'''} \geqslant d_{L_i \cup L'' \cup (L''' + t)} - |t| - \text{const} \geqslant -\frac{1}{2}|t| + \frac{1}{2} d_{L_i \cup L'' \cup (L''' + t)} + \text{const}$), we get (3.13).

From (3.12) and (3.13) we have

$$|N_1(T)| \leqslant D_1 \beta^{(1/2)|t|} \sum_{T': |T'| \leqslant n} \sum_{\{(L_i, L_i')\}} \prod M_1 \beta_1^{d_{L_i \cup L_i'}}$$

where $\beta_1 = \beta^{1/6}$. Now we use the results of Ref. 6.1. From Lemma 2.2 and corollaries from it in this work it follows that

$$\begin{aligned} \|N_1\| &= D_1 \beta^{(1/2)|t|} \sup_{\substack{f \in \mathscr{L}_A^{(1)} \\ \|f\| \leqslant 1}} \sum_{T: 0 \in T} \frac{1}{|T|} \\ &\quad \times \sum_{T': |T'| \leqslant n} \sum_{\{(L_i, L_i')\}} \prod M_1 \beta_1^{d_{L_i \cup L_i'}} |f(T)| \\ &\leqslant \text{const}\, \beta^{(1/2)|t|} \to 0 \qquad \text{as} \quad |t| \to \infty \end{aligned}$$

(b) Nullness of $N_2(T)$ follows from the fact that in (3.9) $\psi(T_1', T_2'+t)\neq 0$ iff $T_1'$ and $T_2'+t$ lie near each other and have no large diameter. But in this case $T_1'\cap T_2'=\emptyset$ and the sum $\sum_{T_1'T_2'}$ in (3.9) is empty.

(c) By Lemma 3.2

$$\begin{aligned}N_3(T)&=\sum_{T_1,T_2}(S_t(\mathscr{T}^{(2)}-\lambda)\,\psi)(T_1,T_2)=(GS_t(\mathscr{T}^{(2)}-\lambda)\,\psi)(T)\\&=\sum_\alpha \chi^{(\alpha)}(T)\end{aligned}$$

where the sum $\sum_\alpha$ is over such volves of $\alpha$ as in (3.7). Here the functions $\chi^{(\alpha)}$ have no nonintersecting supports for different $\alpha$'s but

$$(\chi^{(\alpha)},\chi^{(\beta)})\to 0 \qquad \text{as} \quad |t|\to\infty \qquad \text{for} \quad \alpha\neq\beta$$

(It follows from the finiteness of $\psi$ and exponential decay (2.2) of clustering functions.) From finiteness of $\psi$, boundness of $\mathscr{T}^{(1)}$, and points (a) and (b) of Lemma 3.5 we obtain that $\|N_3\|$ is bounded uniformly in $|t|$. Thus

$$\|N_3\|^2\to\sum_\alpha\|\chi^{(\alpha)}\|^2 \qquad \text{as} \quad |t|\to\infty$$

and the following chain of simple inequalities

$$\begin{aligned}\sum_\alpha\|\chi^{(\alpha)}\|^2_{\mathscr{L}_\Lambda^{(1)}}&=\sum_{T:0\in T}\sum_\alpha\frac{1}{|T|}|\chi^{(\alpha)}(T)|^2\\&=\sum_{\substack{0\in T_1,T_2\neq\emptyset\\ T_1\cap T_2=\emptyset}}\frac{1}{|T_1|+|T_2|}|(S_t(\mathscr{T}^{(2)}-\lambda)\,\psi)(T_1,T_2)|^2\\&\leqslant\|S_t(\mathscr{T}^{(2)}-\lambda)\,\psi\|^2_{\mathscr{L}_\Lambda^{(2)}}\leqslant\varepsilon^2\end{aligned}$$

finishes the proof of Lemma 3.5.

## 4. PROOF OF THE INCLUSION $\sigma_{ess}(\mathscr{T}_\Lambda^{(1)})\subset\sigma(\mathscr{T}_\Lambda^{(2)})$

This inclusion corresponds to the hard part of the proof of the HVZ Theorem (see Refs. 10, 14). We need show that $\sigma(\mathscr{T}_\Lambda^{(1)})\backslash\sigma(\mathscr{T}_\Lambda^{(2)})$ consists only of eigenvalues of finite multiplicity. By Ref. 14 it is equivalent to the fact that for any function $f:\mathbb{R}\to\mathbb{R}$ with compact support not intersecting $\sigma(\mathscr{T}_\Lambda^{(2)})$ the operator $f(\mathscr{T}_\Lambda^{(1)})$ is compact. The idea of the proof of this is in approximating the operator $f(\mathscr{T}_\Lambda^{(1)})$ by evidently compact operators.

In the sequel we use a certain translationally invariant partition of unity $j_\alpha$ in $C^2_{\mathbb{Z}^\nu}=\{T\in C_{\mathbb{Z}^\nu}:|T|\geqslant 2\}$ (analogous to that one used in Ref. 14).

Let

$$|||T||| = \sum_{x,y \in T} |x-y|$$

$$d(T_1, T_2) = \min\{|x-y| : x \in T_1,\ y \in T_2\}$$

The functions $j_\alpha$ of this partition are labeled by the partitions $\alpha$ of the sets $\{1,..., n(\alpha)\}$ into two subsets. Fixing the lexicographic ordering of the subsets of $\mathbb{Z}^\nu$ we have a correspondence between $\alpha$'s and partitions of the subsets $T$ of $\mathbb{Z}^\nu$, $|T| = n(\alpha)$, into two subsets $T_{1\alpha}$ and $T_{2\alpha}$.

**Lemma 4.1.** There exist functions $j_\alpha$ on $C^2_{\mathbb{Z}^\nu}$ such that:

(a) $j_\alpha(T+x) = j_\alpha(T)$, $x \in \mathbb{Z}^\nu$

(b) $\sum j_\alpha = 1$

(c) $j_\alpha(T) = 0$ if $d(T_{1\alpha}, T_{2\alpha}) < |||T||| \cdot |T|^{-3}$ or $|T| \neq n(\alpha)$

*Proof.* (We follow to Ref. 14). We put $j'_\alpha(T) = 0$ if $d(T_{1\alpha}, T_{2\alpha}) < |||T||| \cdot |T|^{-3}$ or $|T| \neq n(\alpha)$ and 1 in other cases. We define $j_\alpha = j'_\alpha(\sum j'_\alpha)^{-1}$. If we show that $\sum j'_\alpha > 0$ then Lemma 4.1 will be proved. Let $T \subset \mathbb{Z}^\nu$, $2 \leqslant |T| < \infty$. We choose $x, y \in T$ such that $|x-y|$ is the greatest possible. Then $r = |x-y| \geqslant |||T||| \cdot |T|^{-2}$. We fix coordinates in $\mathbb{R}^\nu$ such that $x = (0,..., 0)$ and $y = (r, 0,..., 0)$. Consider the domains

$$R_1 = \left\{ z : \frac{r(l-1)}{|T|-1} < z^{(1)} < \frac{rl}{|T|-1} \right\}$$

We see that at least one domain $R_1$ contains no points of $T$. Then the partition $(T_{1\alpha}, T_{2\alpha})$, where $T_{1\alpha}(T_{2\alpha})$ contains points laying in the left (right) side of $R_1$ is the partition, for which $j'_\alpha(T) \neq 0$.

We introduce also the functions $J_{\leqslant K}(T) = 1$ if $|||T||| \leqslant K$ and 0 otherwise; $J_{>K} = 1 - J_{\leqslant K}$. Define the operators

$$F^\alpha : \mathscr{L}^{(1)}_\Lambda \to \mathscr{L}^{(2)}_\Lambda$$

$$\begin{aligned} F^\alpha\varphi(T_1, T_2) &= \varphi(T_1 \cup T_2) \qquad \text{if} \quad T_1 = (T_1 \cup T_2)_{1\alpha} \quad \text{and} \quad T_2 = (T_1 \cup T_2)_{2\alpha} \\ &= 0 \qquad \text{otherwise} \end{aligned} \tag{4.1}$$

The following identity is obvious

$$\begin{aligned} f(\mathscr{T}^{(1)}_\Lambda) &= f(\mathscr{T}^{(1)}_\Lambda)\, J_{\leqslant K} + \sum_\alpha Gf(\mathscr{T}^{(2)}_\Lambda)\, F^\alpha j_\alpha J_{>K} \\ &\quad + \sum_\alpha [f(\mathscr{T}^{(1)}_\Lambda) - Gf(\mathscr{T}^{(2)}_\Lambda)\, F^\alpha]\, j_\alpha J_{>K} \\ &= Q_1 + Q_2 + Q_3 \end{aligned} \tag{4.2}$$

The required assertion follows then from the following:

**Lemma 4.2.**

(a) $Q_1$ is compact

(b) $Q_2=0$

(c) $\|Q_3\| \to 0$ as $K \to \infty$

*Proof.* Part (a) is true since the operator $J_{\leqslant K}$ is compact. Part (b) follows from our choosing of $f$. The point (c) is enough to prove in the cases, when $f(\lambda)=1$ and $f(\lambda)=\lambda$. (This is due to the compactness of $\sigma(\mathscr{T})$.)

Let $f(\lambda)=1$. Then $Gf(\mathscr{T}^{(2)})\,F^\alpha = GF^\alpha$. It is easy to see that the matrix elements of $GF^\alpha$ equal to $(e_T, GF^\alpha e_{T'}) = \delta_{T,T'}\delta_{|T|,n(\alpha)}$. Therefore $Q_3=0$ in this case.

Let $f(\lambda)=\lambda$. Then

$$
\begin{aligned}
&(e_T, (\mathscr{T}^{(1)} - G\mathscr{T}^{(2)}F^\alpha)\, j_\alpha J_{>K} e_{T'}) \\
&= \Bigg( \sum_{\{(L_i, L_i')\}} - \sum_{\substack{\{(L_i,L_i')\}:\\ L_i' \subset T_{1\alpha}' \text{ or } T_{2\alpha}'}} \Bigg) \prod \omega(L_i, L_i')(j_\alpha J_{>K})(T') \\
&= \sum{}' \prod \omega(L_i, L_i')(j_\alpha J_{>K})(T')
\end{aligned}
$$

where the sum $\sum'$ runs over such partitions $\{(L_1, L_i')\}$ of the pair $(T, T')$ that

$$
L_i' \cap T_{1\alpha}' \neq \emptyset \qquad \text{and} \qquad L_i' \cap T_{2\alpha}' \neq \emptyset \qquad \text{for certain } i \tag{4.3}
$$

Next, if $L_i' \cap T_{1\alpha}' \neq \emptyset$ and $L_i' \cap T_{2\alpha}' \neq \emptyset$ then by Lemma 4.1(c)

$$
\begin{aligned}
|\omega(L_i, L_i')|(j_\alpha J_{>K})(T') &\leqslant M_1 \beta^{(1/2)d(T_{1\alpha}', T_{2\alpha}')} \beta^{(1/2)d_{K_i \cup L_i'}} \\
&\leqslant M_1 \beta^{K/2|T'|^3} (\beta_1)^{d_{L_i \cup L_i'}}
\end{aligned} \tag{4.4}
$$

where $\beta_1 = \beta^{1/2}$. Indeed, if $d(T_{1\alpha}', T_{2\alpha}') > K/|T'|^3$ then $d_{\widehat{L_i \cup L_i'}} \geqslant d(T_{1\alpha}', T_{2\alpha}')$ and hence $d_{L_i \cup L_i'} \geqslant \frac{1}{2} d(T_{1\alpha}', T_{2\alpha}') + \frac{1}{2} d_{L_i \cup L_i'}$ and if $d(T_{1\alpha}', T_{2\alpha}') < K/|T'|^3$ then $(j_\alpha J_{>K})(T') = 0$ and (4.4) is also true. From (4.4) we get the estimate

$$
\begin{aligned}
&|(e_T, (\mathscr{T}^{(1)} - G\mathscr{T}^{(2)}F^\alpha)\, j_\alpha J_{>K} e_{T'})| \\
&\leqslant \beta^{K/2|T'|^3} \sum_{\{(L_i, L_i')\}} \prod M_1 \beta_1^{d_{L_i \cup L_i'}}
\end{aligned} \tag{4.5}
$$

Because the number of partitions of the set $T'$ into two subsets equals $2^{|T'|}-1$, it is not difficult to show that (4.5) implies the inequality

$$\left\| P_n \sum_\alpha (\mathscr{T}^{(1)} - G\mathscr{T}^{(2)}F^\alpha)\, j_\alpha J_{>K} P_m \right\| \leqslant \beta^{K/2m^3} 2^m (D\beta_1)^{\max(m,n)}, \qquad D>0$$

where $P_n$ is the orthogonal projection onto $L_n$ (see Corollary 1 from Lemma 2.2 in Ref. 6.I). From this the convergence $\|Q_3\| \to 0$ as $K \to \infty$ can be easily derived. Lemma 4.2 is proved.

## 5. PROOF OF THEOREM 2

The assertion of Theorem 2 is equivalent to the equality $\sigma(D_\Lambda) = \sigma(\mathscr{T}_\Lambda)$ for every $\Lambda \in \mathbb{T}^\nu$.

In order to show the inclusion $\sigma(D_\Lambda) \subset \sigma(\mathscr{T}_\Lambda)$ we observe that by Theorem 1

$$\sigma(\mathscr{T}_\Lambda^{(1)}) = \sigma_d(\mathscr{T}_\Lambda^{(1)}) \cup \sigma_{\text{ess}}(\mathscr{T}_\Lambda^{(1)})$$

where

$$\begin{aligned}\sigma_{\text{ess}}(\mathscr{T}_\Lambda^{(1)}) &= \bigcup_{\Lambda_1+\Lambda_2=\Lambda} \sigma(\mathscr{T}_{\Lambda_1}^{(1)}) \cdot \sigma(\mathscr{T}_{\Lambda_2}^{(1)}) \\ &= \bigcup_{\Lambda_1+\Lambda_2=\Lambda} \sigma_d(\mathscr{T}_{\Lambda_1}^{(1)}) \cdot \sigma_d(\mathscr{T}_{\Lambda_2}^{(1)}) \cup \cdots \\ &= \bigcup_{n\geqslant 2} \bigcup_{\sum \Lambda_i=\Lambda} \prod_1^n \sigma_d(\mathscr{T}_{\Lambda_i}^{(1)}) \cup \cdots\end{aligned}$$

But $\sigma(D_\Lambda) = \{0\} \cup \bigcup_{n\geqslant 0} \bigcup_{\sum \Lambda_i=\Lambda} \prod \sigma_d(\mathscr{T}_{\Lambda_i}^{(1)})$ and from the above our inclusion follows.

Obviously, $0 \in \sigma(D_\Lambda)$. Let now $0<\lambda \in \sigma(\mathscr{T}_\Lambda^{(1)})$. Then either (i) $\lambda \in \sigma_d(\mathscr{T}_\Lambda^{(1)})$ or (ii) $\lambda \in \sigma_{\text{ess}}(\mathscr{T}_\Lambda^{(1)})$. In the case (ii) by Theorem 1 $\lambda \in \sigma(\mathscr{T}_\Lambda^{(2)})$ and hence $\lambda = \lambda_1\lambda_2$, where $\lambda_i \in \sigma(\mathscr{T}_{\Lambda_i}^{(1)})$, $i=1, 2$, $\Lambda_1+\Lambda_2=\Lambda$. Now either $\lambda_1 \in \sigma_d(\mathscr{T}_{\Lambda_1}^{(1)})$ or not. In the second case $\lambda_1 = \lambda_3\lambda_4$, etc. Since each $\lambda_i$ is less than certain constant $\lambda_0<1$ (mass gap) then this procedure finishes at some step. Therefore $\lambda = \lambda_1,\ldots,\lambda_n$, where $\lambda_i \in \sigma_d(\mathscr{T}_{\Lambda_i}^{(1)})$, $\sum \Lambda_i = \Lambda$. This means that $\lambda \in \sigma(D_\Lambda)$. Theorem 2 is complete.

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